ERCIM "ALAIN BENSOUSSAN"
FELLOWSHIP PROGRAMME

ERCIM
European Research Consortium
for Informatics and Mathematics

Scientific Report

First name / Family name	Mads Sielemann / Jakobsen
Nationality	Danish
Name of the *Host Organisation*	NTNU
First Name / family name of the *Scientific Coordinator*	Franz / Luef
Period of the fellowship	01/11/2016 to 31/10/2018

I – SCIENTIFIC ACTIVITY DURING YOUR FELLOWSHIP

My ERCIM postdoc fellowship at NTNU has given me the opportunity to work with Franz Luef on time-frequency analysis, Gabor analysis, and aspects of non-commutative geometry. In particular we have studied the connection between these subjects and used this to answer yet unsolved questions in either theory. I have also worked with Hans G. Feichtinger at the University of Vienna on the advancement of the theory of the Feichtinger algebra. Furthermore, the time as an ERCIM postdoc and its Research Exchange Programme has given me the opportunity to meet, see and listen to a variety of other mathematicians. The time as an ERCIM postdoc has been very educational and enlightening to me and I am very grateful for having had the opportunity to participate in this programme.

II – PUBLICATION(S) DURING YOUR FELLOWSHIP

1. with Hans G. Feichtinger, Distribution Theory by Riemann Integrals, preprint can be found on my personal webpage at NTNU and on arXiv, accepted for publication in the “Indian Journal of Industrial and Applied Mathematics”.

Abstract: It is the purpose of this article to outline a course that can be given to engineers looking for an understandable mathematical description of the foundations of distribution theory and the necessary functional analytic methods. Arguably, these are needed for a deeper understanding of basic questions in signal analysis. Objects such as the Dirac delta and Dirac comb require a proper definition, and it should be possible to explain how one can reconstruct a band-limited function from its samples by means of simple series expansions. It should also be useful for graduate students who want to see how functional analysis can help to understand fairly practical problems, or teachers who want to offer a course related to the "Mathematical Foundations of Signal Processing". The course requires only an understanding of the basic terms from linear functional analysis, namely Banach spaces and their duals, bounded linear operators and a simple version of weak∗-convergence. As a matter of fact we use a set of function spaces which is quite different from the collection of Lebesgue spaces used normally. We thus avoid the use of Lebesgue integration theory. Furthermore we avoid topological vector spaces in the form of the Schwartz space. Although all tools developed and presented can be realized on LCA groups, we restrict our attention in the current presentation to the Euclidean setting, where we have (generalized) functions over Rd. This allows us to make use of simple bounded, uniform partitions of unity, to apply dilation operators and to make use of special functions such as the Gaussian. The problems of the overall current situation, with the separation of theoretical Fourier Analysis as carried out by (pure) mathematicians and Applied Fourier Analysis (as used in engineering applications) are getting bigger and therefore courses filling the gap are in strong need.

2. with Ulrik B. R. Enstad and Franz Luef, Time-frequency analysis on the adeles over the rationals, preprint can be found on my personal webpage at NTNU and on arXiv, the paper is submitted for possible publication in a peer reviewed journal.
   Abstract: We show that the construction of Gabor frames in L2(R) with generators in S0(R) and with respect to time-frequency shifts from a rectangular lattice $\alpha Z \times \beta Z$ is equivalent to the construction of certain Gabor frames for L2 over the adeles over the rationals and the group R×Qp. Furthermore, we detail the connection between the construction of Gabor frames on the adeles and on R×Qp with the construction of certain Heisenberg modules.

3. with Franz Luef, Duality of Gabor frames and Heisenberg modules, preprint can be found on my personal webpage at NTNU and on arXiv, the paper is submitted for possible publication in a peer reviewed journal.
   Abstract: Given a locally compact abelian group G and a closed subgroup Λ in $G \times \hat{G}$, Rieffel associated to Λ a Hilbert C∗-module E, known as a Heisenberg module. He proved that E is an equivalence bimodule between the twisted group C∗-algebra $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, c^{--})$, where Λ° denotes the adjoint subgroup of Λ. Our main goal is to study Heisenberg modules using tools from time-frequency analysis and pointing out that Heisenberg modules provide the natural setting of the duality theory of Gabor systems. More concretely, we show that the Feichtinger algebra S0(G) is an equivalence bimodule between the Banach subalgebras $S0(\Lambda, c)$ and $S0(\Lambda^\circ, c^{--})$ of $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, c^{--})$, respectively. Further, we prove that S0(G) is finitely generated and projective exactly for co-compact closed subgroups Λ. In this case the generators g1,...,gn of the left $S0(\Lambda)$-module S0(G) are the Gabor atoms of a multi-window Gabor frame for L2(G). We prove that this is equivalent to g1,...,gn being a Gabor super frame for the closed subspace generated by the Gabor system for Λ°. This duality principle is of independent

interest and is also studied for infinitely many Gabor atoms. We also show that for any non-rational lattice Λ in R2m with volume s(Λ)<1 there exists a Gabor frame generated by a single atom in S0(Rm).

4. with Hans G. Feichtinger, The inner kernel theorem for a certain Segal algebra, preprint can be found on my personal webpage at NTNU and on arXiv, the paper is submitted for possible publication in a peer reviewed journal.
   Abstract: The Segal algebra S0(G) is well defined for arbitrary locally compact Abelian Hausdorff (LCA) groups G. Despite the fact that it is a Banach space it is possible to derive a kernel theorem similar to the Schwartz kernel theorem, of course without making use of the Schwartz kernel theorem. First we characterize the bounded linear operators from S0(G1) to S′ 0(G2) by distributions in S′ 0(G1×G2). We call this the "outer kernel theorem". The "inner kernel theorem" is concerned with the characterization of those linear operators which have kernels in the subspace S0(G1×G2), the main subject of this manuscript. We provide a description of such operators as regularizing operators in our context, mapping S′ 0(G1) into test functions in S0(G2), in a w∗-to norm continuous manner. The presentation provides a detailed functional analytic treatment of the situation and applies to the case of general LCA groups, without recurrence to the use of so-called Wilson bases, which have been used for the case of elementary LCA groups. The approach is then used in order to describe natural laws of composition which imitate the composition of linear mappings via matrix multiplications, now in a continuous setting. We use here that in a suitable (weak) form these operators approximate general operators. We also provide an explanation and mathematical justification used by engineers explaining in which sense pure frequencies "integrate" to a Dirac delta distribution.

5. with Ludwik Dabrowski, Giovanni Landi and Franz Luef: Solitons of general topological charge over noncommutative tori, preprint can be found on my personal webpage at NTNU and on arXiv, the paper is submitted for possible publication in a peer reviewed journal.
   Abstract: We continue the study of solitons over noncommutative tori from the perspective of time-frequency analysis and treat the case of a general topological charge. Solutions are associated with vector bundles of higher rank over noncommutative tori. We express these vector bundles in terms of vector-valued Gabor frames and apply the duality theory of Gabor analysis to show that Gaussians are solitons of general topological charge over noncommutative tori. An energy functional for projections over noncommutative tori is the basis for the self and anti-self duality equations of the solitons which turns out to have a reformulation in terms of Gabor atoms and we prove that projections generated by Gaussians minimize this energy functional. Finally, we comment on the case of the Moyal plane and the associated continuous vector-valued Gabor frames and show that Gaussians are the only class of solitons.

6. On a (No Longer) New Segal Algebra: A Review of the Feichtinger Algebra, published in J. Fourier Anal. Appl, 2018.
   Abstract: Since its invention in 1979 the Feichtinger algebra has become a useful Banach space of functions with applications in time-frequency analysis, the theory of pseudo-differential operators and several other topics. It is easily defined on locally compact Abelian groups and, in comparison with the Schwartz(-Bruhat) space, the

Feichtinger algebra allows for more general results with easier proofs. This review paper develops the theory of Feichtinger's algebra in a linear and comprehensive way. The material gives an entry point into the subject and it will also bring new insight to the expert. A further goal of this paper is to show the equivalence of the many different characterizations of the Feichtinger algebra known in the literature. This task naturally guides the paper through basic properties of functions that belong to this space, over operators on it, and to aspects of its dual space. Additional results include a seemingly forgotten theorem by Reiter on Banach space isomorphisms of the Feichtinger algebra, a new identification of Feichtinger's algebra as the unique Banach space in $L1$ with certain properties, and the kernel theorem for the Feichtinger algebra. A historical description of the development of the theory, its applications, and a list of related function space constructions is included.

7. with Yavar Khedmati, Approximately dual and perturbation results for generalized translation-invariant frames on LCA groups, published in Int. J. Wavelets Multiresolt. Inf. Process, 2017.
   Abstract: The results in this paper can be divided into three parts. First, we generalize the recent results of Benavente, Christensen and Zakowicz on approximately dual generalized shift-invariant frames on the real line to generalized translation-invariant (GTI) systems on locally compact abelian (LCA) groups. Second, we explain in detail how GTI frames can be realized as g-frames. Finally, the known results on perturbation of g-frames and results on perturbation of generalized shift-invariant systems are applied and extended to GTI systems on LCA groups.

III – ATTENDED SEMINARS, WORKHOPS, CONFERENCES

- Seminar: I have attended the regular Analysis Seminar at the Mathematics Department at NTNU, where I also have given talks myself
- Conference: Strobl18 - Harmonic Analysis and Applications, June 3rd - 8th, 2018, Strobl, Austria.
- Conference: Quantum Harmonic Analysis and Symplectic Geometry, April 21st - 24th. 2018, Strobl, Austria.
- Conference: Meeting of the Catalan, Spanish, Swedish Math Societies, June 12th - 15th, 2017, Linköping University in Umeå, Sweden.
- Conference: Aspects of Time-Frequency Analysis conference, June 5th - 7th, 2017, at the Politecnico di Torino, Italy.

IV – RESEARCH EXCHANGE PROGRAMME (REP)

For my Research Exchange Programme I visited the Numerical Harmonic Analysis Group (NuHAG) at the University of Vienna in Austria for a few months. This stay has given me the opportunity to meet several other researchers, post docs and PhDs that are active in my research area. I am very grateful for the opportunity to spend time in Vienna and it has been a positive experience academically as well as personally.