ERCIM "ALAIN BENSOUSSAN"
FELLOWSHIP PROGRAMME

Scientific Report

First name / Family name	Hung / Le
Nationality	Vietnam
Name of the *Host Organisation*	Norwegian University of Science and Technology
First Name / family name of the *Scientific Coordinator*	Mats / Ehrnström
Period of the fellowship	01/11/2019 to 31/10/2020

I – SCIENTIFIC ACTIVITY DURING YOUR FELLOWSHIP

During the fellowship, I was working on four projects and gave a presentation in NTNU Department of Mathematical Sciences seminar. I also attended weekly seminars in the department.

1) Solitary waves to Whitham-type equations:
   We consider a nonlocal equation in the moving frame with wave speed $\mu > 0$:
   $$-\mu\varphi + \varphi^p + K * \varphi = 0,$$
   where p > 1 is a real number, and the convolution kernel K in the main equation is assumed to satisfy certain conditions. In particular, the Fourier transform $\widehat{K}$ is smooth, even, has its maximum at the origin and has algebraic decay. That is, $L = K *$ is a smoothing operator. Moreover, the fact that K is positive implies that K is concave at the origin. The main novelty here is to adapt the method described in the paper by Arnesen, Ehrnström, and Stefanov but in the Orlicz space. This approach has the advantage that it does not rely on smallness. This paper generalizes the results for the main equation to all equations satisfying the assumptions on the convolution kernel.

2) Waves of maximal height for a class of nonlocal equations with inhomogeneous symbols:
Our object of study is the *fractional Korteweg--de Vries equation*

$$u_t + Lu_x + uu_x = 0,$$

where L denotes the Fourier multiplier operator with symbol $m(k) = (1 + k^2)^{-\alpha/2}$. We are looking for a 2π-periodic traveling-wave solutions. My approach is to combine the analysis from two papers. In the paper by Ehrnström and Wahlén (2019), the investigation started with the Whitham equation, where convolution kernel L is the Fourier multiplier operator given by the inhomogeneous symbol $m(k) = \sqrt{\frac{\tanh k}{k}}$, which is of order greater than -1. On the other hand, Bruell and Dhara (2018) considered the fractional KdV, where L is the Fourier multiplier operator given by the homogeneous symbol $m(k) = |k|^{-r}, r > 1$, which is of order less than -1.

The main novelty here is to implement the approach used in both papers when the symbol is a Bessel potential function, inhomogeneous-type and order less than -1. The paper starts with introducing functional-analytic settings and discussion about the operator L, and then provides some a priori properties of an even solution to the main equation. The heart of the analysis is the local and global bifurcation. The resulting wave is proved to be exactly Lipschitz.

3) Waves under the effects of gravity, surface tension, and vorticity:
Analytic global bifurcation theory is used to construct a large variety of families of steady periodic two-dimensional capillary-gravity water waves with real-analytic vorticity distributions, propagating in an incompressible fluid. The waves that are constructed can possess overhanging shape. Ideas are taken from the paper "Gravity-perturbed Crapper waves" by Akers, Ambrose, and Wright, the paper "Trimodal of steady water waves" by Ehrnström and Wahlén, and the paper "Traveling gravity water waves with critical layers" by Aasen and Varholm. Because of the high-order of complexity of the problem with all three parameters: gravity, surface tension, and vorticity, the problem is in the investigation stage.

4) Numerics for point vortex waves:
We plan to use computer simulations to verify the stability results from the previous papers. In particular, we hope to show that the waves generated by a single point vortex are stable, while waves generated a pair of point vortices (finite dipole) are unstable. We also planned to extend to the point dipole case (more singularity). The codes are written in FORTRAN language. Thanks to the support of NTNU, codes for a single point vortex ran successfully. Before extending it to a finite dipole and point dipole cases, we need to optimize the codes to deal with more complicated computation.

5) I, with the help of my mentor Ehrnström, prepared applications for “Mobility Grant” and “Marie-Curie Fellowship”. This is to try to secure a postdoctoral position after the fellowship.

II – PUBLICATION(S) DURING YOUR FELLOWSHIP

No paper has been published yet. There are two papers in the final stage before submitting.

[1] Mathias Nikolai Arnesen and Hung Le. Solitary waves to whitham-type equations. In preparation, 2020.

Abstract: Using constrained maximization techniques, we construct soli- tary waves for a class of weakly dispersive non-local equations. The waves are bell-shaped in the sense that they are even, one-sided monotone, and attain their maximum at the origin. The method is based on constrained maximization in an appropriate (Orlicz) function space tailored to the equation.

[2] Hung Le. Waves of maximal height for a class of nonlocal equations with inhomogeneous symbols. In preparation, 2020.

Abstract: In this project, we show the existence and regularity of 2π-periodic traveling-wave solutions of a class of nonlocal equations with Bessel potential symbol of order α, where $\alpha > 1$. The regularity of the highest wave is proved to be exactly Lipschitz.

III – ATTENDED SEMINARS, WORKHOPS, CONFERENCES

03/12/2019, “On the existence and instability of solitary water waves with a finite dipole”. Differential Equations and Numerical Analysis (DNA) group at NTNU.

I also attended weekly seminars at NTNU, online seminars at the University of Missouri, USA, and online conferences (October 2020) such as talks at Fields Institute. No presentations were given.

Due to the COVID-19 pandemic, the SIAM conference on Nonlinear Waves and Coherent Structures (NWCS20), which I had planned to give a presentation in July 2020, was cancelled.

IV – RESEARCH EXCHANGE PROGRAMME (REP)

I did the REP with Professor Emmanouil Milakis from the Department of Mathematics and Statistics at the University of Cyprus in October 2020. Milakis research interest is in the area of partial differential equations and free boundary problems. Recently he has been working on parabolic equations with various types of boundary conditions, which is

very similar to my previous project on elliptic equations. During this time, we met through online Zoom meetings. I presented my current projects that I was doing for the postdoctoral position at NTNU such as the set-up of the problem, the approach, and the main difficulties.