SCIENTIFIC REPORT

ERCIM ALAIN BENSOUSSAN FELLOWSHIP PROGRAM

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Period of the fellowship	01.10.2019 – 30.09.2020

1. Scientific activity during your fellowship

1.1. **Background on non-commutative probability theory.**
My mathematical research addresses problems and questions is in the general area of non-commutative probability theory.

Researchers in non-commutative probability theory have developed an algebraic point of view on the theory of probability, both classical and non-commutative. It all begins with a redefinition of basic objects. For probabilists, a measured space $(\Omega, \mathcal{A}, \mathbb{P})$ is the root of any further advanced setting.The concept of non-abelanisation states that to construct a non-commutative analogue of classical probability theory, any measured space should first be associated with a commutative algebra and exhibit properties that with commutativity characterize these algebras. With a given measured space is associated the algebra of essentially bounded random variables; these being the only commutative Von Neumann algebras. Hence, in non-commutative probability theory, Von Neumann algebras are considered as basic objects in the same way measured spaces are basic objects in classical probability theory. Many concepts of classical probability find their counterpart in non-commutative settings, such as random variables and the notion of independence. A sub-theory of non-commutative probability theory is named free probability theory, which has been shown to be efficient in constructing a framework for studying large random matrices. As mentioned, many notions in classical probability find their counterparts in non-commutative probability, in particular *conditional expectation* and related properties of stochastic processes (such as martingale properties, Markov properties, etc.) can be immediately stated for paths of operators in a Von Neumann algebra. Conditional expectation associates to a random variable another one, measurable with respect to a smaller σ-field. In the language of Von Neumann algebras, conditional expectation is a projection onto a smaller algebra. Conditional notions (such a conditional independence) find counterparts in the theory of free *operator-valued probability* (or *amalgamated free probability*). Further below, in Section 1.2.1, we describe a line of research in the realm of operator-valued free probability. To fix notations, an operator-valued probability space is an algebra $\mathcal{A}$ on which a non-commutative algebra B acts on the left and on the right in compatible ways. The moments of a random variable is a sequence of elements in the algebra B, computed by mean of a bimodule map $E : \mathcal{A} \to B$ (in scalar probabiliy theory $B = \mathbb{C}$).

1.2. **Operadic perspective on non-commutative probability theory.**
Hopf algebras are ubiquitous, in non-commutative probability theory they are state spaces for non-commutative processes. Since the work of K. Ebrahimi-Fard and F. Patras, they gained a new role as the underlying algebraic structures, together with unshuffle bialgebras, implementing the relation between moments and cumulants in free, boolean and monotone non-commutative probability theory. In this project, we extended their original work to the operator-valued settings.

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1.2.1. *Shuffle group laws in operator-valued free probability.* In classical probability, moments and classical cumulants of a random variable relate to each other using a genuinely defined convolution algebra, commonly refered to as the *Möbius* convolution algebra on the poset of partitions. Remarkably, if we replace the poset of partitions by the poset of non-crossing partitions a similar connection holds between moments and free cumulants of a random variable. In non-commutative probability theory, the moments of random variable and the free cumulants are extended as *multiplicative functions on the poset of non-crossing partitions*, both can be computed on a partition by first evaluating them on each block of the partition and then multiplying alltogether the results. Notice that in this perspective moments and cumulants are on the same footings. The free moment-cumulant relation reads

$$m_n(a) = \sum_{\pi \in \mathrm{NC}(n)} k_\pi(a) \tag{MC}$$

The relation (MC) is formally equivalent to a fixed point equation for the generating functions of the sequence of moments $(m_n(a))_{n\geq 1}$ and the generating functions of the sequence of cumulants, the so-called R transform. In [5, 3, 4, 2] K. Ebrahimi-Fard, L. Foissy, J. Koch and F. Patras proposed an original Lie theoretical perspective on moments and free cumulants based on the fundational notion of Shuffle Hopf algebras (or dendrimorphic algebras). *To moments and free cumulants of random variables corresponds, respectively, a morphism and an infinitesimal morphism* on a bialgebra (H, Δ, ε) of words on words on random variables. The authors in [5, 3] prove that the class of characters on H is a Lie group, the product $(\star)$of two characters is well-defined and the result is a character on H. Remarkably, *this convolution product splits as a sum of two bilinear non-associative products, the left and right half-shuffle.* These two composition laws are commonly denoted $\prec$ and $\succ$ and satisfy the axioms of a shuffle algebra. To each of these products corresponds an exponential map from the Lie algebra of infinitesimal characters to the group of characters. The left half-shuffle exponential of the free cumulant infinitesimal morphism is equal to the moment map. Hence, *the left half-shuffle implements the moment-cumulant relation* (MC). The right half-shuffle exponential and the shuffle exponential are respectively, interpreted in the framework of boolean and monotone probability.

The poset of non-crossing partitions, so fundamental in the earlier formulation of free probability theory, is blurred in the shuffle approach. In [2], the authors unveil the connection between the half-shuffle products $\prec$ and $\succ$ defined earlier, on one hand, and an *operadic structure on non-crossing partitions* on the other hand which formalizes the idea of insertion of a sequence of non-crossing partitions into another non-crossing partition.

The values of the function on non-crossing partitions extending the sequence moments (respectively, the sequence of free cumulants) of a random-variables, are values of a character M (respectively, K) on the algebra H^{NC} of non-commutative polynomials on non-crossing partitions.

A general algebraic construction applied to the gap-insertion operad endows the space H^{NC} with a bialgebra structure. The coproduct splits as two half-shuffle coproducts. *Incidentely the class of characters of H^{NC} is a shuffle algebra. Besides, M is a left-half shuffle exponential.*

In the operator-valued case, the relation (MC) holds if m and k are replaced by their operator-valued counterpart. These two functions on non-crossing partitions extending the sequences of moments and free cumulants are much more difficult to describe, since the linear order of the random variables has to be maintained and the moments and cumulants are now B-valued, see [6]. The starting point of our work is the following result.

▶ We prove that the algebraic object encoding *moments and free cumulants of a random variable* in free operator-valued probability theory is an *operadic morphism* on the gap-insertion operad of non-crossing partitions valued in the operad of multilinear maps on B.

These operadics morphisms extends to algebra morphisms M^{op} and K^{op} on the space H^{NC}. These morphisms are in fact compatible with both the concatenation law on H^{NC} and the gap-insertion operadic structure. *Such morphisms are called properadic morphisms.*

Notice that combinatorics of boolean probability theory involve the collection of interval partitions, which is *not* naturally a sub-operad of the gap-insertion operad. Hence, we can not require compatibility of boolean cumulants with respect to the gap-insertion composition law.

We quickly reckon that words on non-crossing partitions are to be seen as operators with multiple inputs and multiple outputs, one for each partition in the word. This leads us to consider generalizations of the notion of *operads for operators with multiple inputs and outputs*. First introduced in [7] by B. Vallette, these generalizations are called PROS and properads. To the operation of concatenation corresponds an horizontal monoidal tensor product and branching outputs of operators to inputs of others corresponds a vertical monoidal structure (an operadic compostion in the single output case).

As a matter of facts, we use a much simpler and planar versions of these monoidal structures on bicollections. As a consequence, the horizontal and vertical tensor products in our case enjoy a property that is known as the *lax property* and turns to be the cornerstone of our Lie theoretic perspective on operator-valued probability theory. In category theory, Lax-compatible vertical and horizontal tensor products are closely related to the notion of 2-*monoidal categories*, as introduced by Aguiar, see for example [1].

▶ In table 1 we have collected the compatibilities between the moments and free, boolean and monotone cumulants with respect to the two monoidal structures.

Non-crossing partitions	**Scalar**	**Operator-valued**
Free M/C.	Alg. mor.	Horizontal mor., Vertical mor. (Properad mor.)
Boolean M/C.	Alg. mor.	Horizontal mor., inverse of a Vert. mor. (Boolean mor.)
Monotone M/C.	Alg. mor.	Horizontal mor.

FIGURE 1. Comparison between properties of morphisms implementing cumulants and moments in the shuffle approach of scalar and operator-valued probability theory.

▶ We introduce the notion of unshuffle bi-algebras in the context of 2-monoidal categories (we use the terminology $\boxtimes\otimes$-bialgebra) and prove that the space of morphisms from this bi-algebra to a properad of endomorphisms is a shuffle algebra.

In Table 2 we have collected the structural morphisms of a $\boxtimes\otimes$- Hopf algebra, in the right column and the ones for a Hopf algebra in left column. Using the terminology of 2-monoidal category theory, $(H^{\rm NC}, m^{\otimes}, \Delta)$ is bimonoid and $(H^{\rm NC}, m^{\otimes}, m^{\boxtimes})$ is a dimonoid.

Proposition. *The properad of words on non-crossing partition is a* $\boxtimes\otimes$*-Hopf algebra. As a consequence, the class of morphisms from the bicollection* H^{NC} *to the bicollection of multilinear maps on* B *is a shuffle algebra.*

	Scalar	Operator-valued
Coproduct map	$\Delta : H^{\rm NC} \to H^{\rm NC} \otimes H^{\rm NC}$	$\Delta : H^{\rm NC} \to H^{\rm NC} \boxtimes H^{\rm NC}$
Product map	$m^{\otimes} : H^{\rm NC} \otimes H^{\rm NC} \to H^{\rm NC}$	$m^{\boxtimes} : H^{\rm NC} \boxtimes H^{\rm NC} \to H^{\rm NC}$ $m^{\otimes} : H^{\rm NC} \otimes H^{\rm NC} \to H^{\rm NC}$
Antipode	?	$S = (-1)^{\pi}\delta_{\pi\in{\rm Int}}\pi$

FIGURE 2. Structural morphisms on $H^{\rm NC}$ in the scalar case and in the operator-valued case.

This last proposition allows for a complete extension of the shuffle perspective on free probability to the operator-valued case.

Proposition. *A bicollection map from the bicollection* $H^{\rm NC}$ *to the bicollection of multilinear maps on* B *is a properadic morphism if and only if it is the left half-shuffle exponential of an infinitesimal morphism (satisfying a mild assumption).*

▶ In particular moments and free cumulants of random variables, as operadic morphisms M^{op} and K^{op} are solution of two left half-shuffle fixed point equations:

$$K^{op} = \eta \circ \varepsilon + k \prec K^{op}, \quad M^{op} = \eta \circ \varepsilon + m \prec M^{op}. \tag{1}$$

To retrieve the half-suffle fixed point equation that stands for the free moment-cumulant relation, we define a second $\boxtimes \otimes$ unshuffle Hopf algebra H^W obtained from a operadic structure on words on random variables, paralleling the construction for non-crossing partitions.

▶ There exists a map, called the *splitting morphism*, from H^W to H^{NC} compatible with the two $\boxtimes \otimes$-unsuffle bialgebra structure.

Proposition. *The free operator-valued moment-cumulant relation is equivalent to a the left half-shuffle fixed point equation*

$$M = \eta \circ \varepsilon + k \prec M. \tag{2}$$

where M and k are, respectively, the pullback by the splitting map of the properadic morphism M^{op} on H^{NC} implementing moments, respectively, the infinitsimal morphism k on H^{NC}.

2. Publication during your fellowship

Publication	Status
Shuffle approach to operator-valued probability theory, arXiv:2005.12049	Submitted to Advances in Mathematics

Shuffle approach to operator-valued probability theory, Gilliers Nicolas, 2020

We extend the shuffle algebra perspective on scalar-valued non-commutative probability theory to the operator-valued case. We start by associating to the operator-valued distribution and free cumulants of a random variable elements of the homomorphism operad corresponding to the algebra acting on the operator-valued probability space. Using notions coming from higher category theory, we are able to define an unshuffle Hopf algebra like structure on a PROS of non-crossing partitions. We do the exact same construction for a PROS of word insertions and construct an unshuffle morphism, the splitting map, between the two unshuffle Hopf algebras. We obtain two half-shuffle fixed point equations corresponding to, respectively, free and boolean moment-cumulants relations.

3. Attended seminars, workshops, conferences

Conference	Location	Date	Remarks
Young researchers between geometry and stochastic analysis	Bergen, Norway	12-14.02.2020	
Manifolds and Geometric Integration Colloquia	Trysil, Norway	23-27.03.2020	Cancelled
Algebraic structure in Perturbative Quantum Field Theory	Paris, France	27.01-31.01.2020	Postponed
Conference in honours of Speicher 60th birthday	Saarbrucken, Germany	8.06-12.06.2020	Cancelled
ACPMS online seminar	Online, NTNU, Trondheim, Norway	Fridays	Speaker
Voiculescu's online free probability seminar	Online, University of california, Berkeley	Mondays	Speaker
Applied category theory online confernece	Online, MIT, Cambridge, US	06.07-10.07.2020	
Geometry of curves in time-series analysis	Online, MPI Leipzig, Germany	11.08-14.08.2020	

4. Research exchange program

None.

References

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[2] Kurusch Ebrahimi-Fard, Loïc Foissy, Joachim Kock, and Frédéric Patras. Operads of (noncrossing) partitions, interacting bialgebras, and moment-cumulant relations. *arXiv preprint arXiv:1907.01190*, 2019.

[3] Kurusch Ebrahimi-Fard and Frédéric Patras. Cumulants, free cumulants and half-shuffles. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2176):20140843, 2015.

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[7] Bruno Vallette. A Koszul duality for props. *Transactions of the American Mathematical Society*, 359(10):4865–4943, 2007.