1. Information on candidate

First name/family name	Susovan PAL
Nationality	Indian
Name of the Host Organization	INRIA
First name/family name of the scientific co-ordinator	Olivier Colliot
Period of the fellowship	01/10/2014-30/09/2015

2. Scientific activity during fellowship

My main research so far is to 1)reintroduce a numerical algorithm for computing parallel transport for any general Riemannian manifold, and compute its optimal error bounds, and 2)use the parallel transport to define a mixed effect longitudinal model measuring the progress of Alzheimer's disease. The use of parallel transport in statistical models is of particular importance in medical imaging and other related fields.

2.1. **Numerical algorithm for parallel transport on Riemannian manifolds.** Let $P_{0,s}(w)$ be the parallel translate along a geodesic γ of a vector w, and let $J(s)$ denote the Jacobi field along γ. Motivated by the equation $\left\| P_{0,s}(w) - \frac{J(s)}{s} \right\| = o(s)$, we define the following algorithm to approximate parallel transport $P_{0,s}(w)$ as follows: we divide the interval $[0,1]$ into N subintervals $[\frac{k}{N}, \frac{k+1}{N}], 0 \leq k \leq N-1$. then we inductively define $\{J_k\}_{0 \leq k \leq N-1}$ by:

$J_0 = J$ and for $k \geq 1, J_k(\frac{k}{N}) = 0, J'_k(\frac{k}{N}) = \frac{J_{k-1}(\frac{k}{N})}{\frac{1}{N}}$.

As for the estimated parallel transport, we define it by:

$\hat{P}_{0,\frac{k}{N}}(w) := J'_k(\frac{k}{N}) = \frac{J_{k-1}(\frac{k}{N})}{\frac{1}{N}}$. So, in particular, the estimated time 1 parallel transport of w is $\hat{P}_{0,\frac{N}{N}}(w)$. We denote this vector by $\hat{P}^{(N)}_{0,1}(w)$.

We theoretically proved the following **convergence theorem**:

Theorem 1. $\left\| \hat{P}^{(N)}_{0,1}(w) - P_{0,1}(w) \right\| \leq C(metric).\frac{1}{N}.\|w\|$, *where* $C(metric)$ *is a positive constant depending only on the underlying Riemannian metric.*

We also showed that the $O(\frac{1}{N})$ bound in theorem 1 is indeed optimal, in the sense that we cannot replace it by $O(f(N))$ where $f(N) \leq \frac{1}{N}$, for example, by $f(N) = \frac{1}{N^2}$. As we will see in the next lemma, the sphere and the hyperbolic spaces provide such examples.

Lemma 1. *Let* M *denote either unit sphere* $\mathbb{S}^n$ *or the hyperbolic space* $\mathbb{H}^n$. *Then, there exist positive constants* C_1, C_2 *such that* $\left\| \hat{P}^N_{0,1}(w) - P_{0,1}(w) \right\| \geq C.\frac{1}{N}$ *for every geodesic* γ. *Here* $\hat{P}^N_{0,1}(w), P_{0,1}(w)$ *denote the estimated and actual parallel transport along* γ.

We performed numerical experiments with spheres and plotted the errors of estimation, with the error plot shown on the next page. This backs up our theoretical calculation.

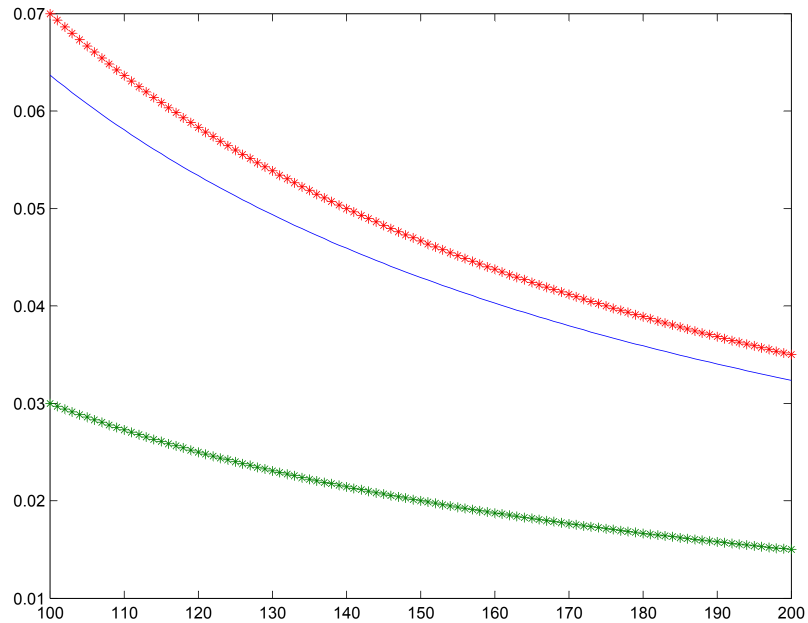

FIGURE 1. Plot of error in approximating parallel transport

The above figure plots the error for computing parallel transport for $\mathbb{S}^2$ by using this algorithm $\left\|\hat{P}_{0,1}^N(w) - P_{0,1}(w)\right\|$ for $100 \leq N \leq 200$. We observe that the error is bounded between the graphs of $\frac{3}{N}$ and $\frac{7}{N}$.

2.2. **A mixed effect longitudinal model for Alzheimer's disease progression.** We measure the i-th patient n_i number of times, and define the j-th observation for the $i-$the patient to be y_{ij}. Our proposed mixed effect longitudinal model uses parallel transport, and is defined by:

$$y_{ij} = P^c_{W_i}(t_{ij}) + \epsilon_{ij}$$

where, we impose the following distributional assumptions:
1)c is mean trajectory: equivalent of $\beta_1 + \beta_2 t$, and c is assumed to be geodesic, 2)W_i is orthogonal to $c'(0)$, 3)Fixed effects: Geodesic c, i.e. $c(0), c'(0)$, 4)Random effects: $W_i \sim \mathcal{N}(0, R)$, 5)Errors:$\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.

The next goal will be to estimate the fixed and the random effect parameters in this model.

3. Publications during fellowship

Stanley Durrleman and Susovan Pal, Approximating parallel transport using Jacobi fields and a model in computational anatomy: *in preparation*.

4. Attended seminars, workshops, conferences

Infinite dimensional Riemannian Geometry and its application to Computational Anatomy, week 6: *Erwin Schrdinger Institute for Mathematical Physics, Vienna, Austria, February 15-21, 2015.*

5. Research Exchange Program

For the research exchange program, I went to Dept. of Mathematics, University of Vienna, Austria and had discussions with Prof. Peter Michor and his group members, who were also the organizers of the conference I attended, and work in more mathematical part of medical imaging. Part of this discussion were done in February 2015, during the conference itself and the rest took place in September 2 and 3, 2015.

The question of interest here is to characterize the following functions: $F_v = \phi_1^v - Id$, where $\phi_1^v \in G_V$, a certain subgroup of diffeomorphim group used in LDDMM framework of computational anatomy. The motivation is: if ϕ_1^v is a slight perturbation of Id mapping, what can we say about the perturbation? For V=Sobolev space $H^k(\Omega, R^d), F_v$ is also in $H^k(\Omega, R^d)$. However, when V is the RKHS with Gaussian kernel or any other kernel, similar result does not hold, and the question remains open.